

GLOBALLY F-REGULAR F -SANDWICHES OF DEGREE p OF A PROJECTIVE SPACE

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ABSTRACT. We prove that globally F-regular F -sandwiches of degree p of a projective space are toric varieties.

INTRODUCTION

We work over an algebraically closed field k of positive characteristic p . Let X be a variety over k . If an iterated Frobenius morphism $F^e : X \rightarrow X$ factors as $X \rightarrow Y \rightarrow X$, we say Y is a Frobenius sandwich of X . If $e = 1$, Frobenius sandwich Y is called an F -sandwich. Given a variety, it is natural to ask what kinds of varieties appear as Frobenius sandwiches. From the view point of Frobenius splitting, we have considered the following problem in [4]:

Problem. Given a globally F-regular variety X , classify globally F-regular Frobenius sandwiches of X .

In [4] and [8], we have classified globally F-regular F -sandwiches of the projective plane and Hirzebruch surfaces. F -sandwiches are constructed by glueing quotients of affine patches by a rational vector field. The classifications have been achieved by explicit calculations of coordinate changes. In this paper, we consider globally F-regular F -sandwiches of a projective space \mathbb{P}^n . The following is the main result of this paper.

Theorem. *Globally F-regular F -sandwiches of degree p of \mathbb{P}^n are toric varieties.*

For the proof, we give a description of F -sandwiches as Proj of the constant ring of the homogeneous coordinate ring of \mathbb{P}^n by a global section of $T_{\mathbb{P}^n}$. Using that description, we show that globally F-regular F -sandwiches of degree p of \mathbb{P}^n are toric varieties without tedious calculations of coordinate changes.

In Section 1, we review generalities on Frobenius sandwiches and globally F-regular varieties. In Section 2, we give a description of global sections of tangent bundle $T_{\mathbb{P}^n}$ as a derivation over the homogeneous

coordinate ring of \mathbb{P}^n . In Section 3, we give the proof of the main theorem.

1. FROBENIUS SANDWICHES AND GLOBALLY F-REGULAR VARIETIES

In what follows, we do not distinguish the absolute Frobenius morphism and the relative one, since we work over the algebraically closed field. See [5] for the definitions of those Frobenius morphisms.

First we review generalities on Frobenius sandwiches.

Definition 1.1 ([4], [3]). Let X be a smooth variety over k . A normal variety Y is an F^e -sandwich of X if the e -th iterated relative Frobenius morphism of X factors as

$$\begin{array}{ccc} X & \xrightarrow{F_{\text{rel}}^e} & X^{(-e)} \\ & \searrow \pi & \nearrow \rho \\ & Y & \end{array}$$

for some finite k -morphisms $\pi : X \rightarrow Y$ and $\rho : Y \rightarrow X^{(-e)}$, which are homeomorphisms in the Zariski topology. The Frobenius sandwich Y is of *degree p* if the degree of the morphism $\pi : X \rightarrow Y$ is p .

A 1-*foliation* of X is a saturated p -closed subsheaf L of the tangent bundle T_X closed under Lie brackets, where L is said to be p -closed if it is closed under p -times iterated composite of differential operators.

Let X be a smooth variety over k , and $K(X)$ be the function field of X . A rational vector field $\delta \in \text{Der}_k K(X)$ is p -closed if $\delta^p = \alpha\delta$ for some $\alpha \in K(X)$. Then there are one-to-one correspondences among the followings:

- F -sandwiches of degree p of X ;
- invertible 1-foliations of X ;
- p -closed rational vector fields of X modulo an equivalence \sim .

(1) Rational vector fields and F -sandwiches: Let $\delta, \delta' \in \text{Der}_k K(X)$. We denote $\delta \sim \delta'$ if there exists a non-zero rational function $\alpha \in K(X)$ such that $\delta = \alpha\delta'$. We can easily check that \sim is an equivalence relation between rational vector fields. Let $\{U_i = \text{Spec } R_i\}_i$ be an affine open covering of X . Given a p -closed rational vector field $\delta \in \text{Der}_k K(X)$, we have a quotient variety X/δ defined by glueing $\text{Spec } R_i^\delta$, where $R_i^\delta = \{r \in R_i \mid \delta(r) = 0\}$, and a quotient map $\pi_\delta : X \rightarrow X/\delta$ induced from the inclusions $R_i^\delta \subset R_i$. Then X/δ is an F -sandwich of degree p with the finite morphism $\pi_\delta : X \rightarrow X/\delta$.

(2) Rational vector fields and 1-foliations: A rational vector field $\delta \in \text{Der}_k K(X)$ is locally expressed as $\alpha \sum f_i \partial / \partial s_i$, where s_i are local coordinates, f_i are regular functions without common factors, and $\alpha \in K(X)$. The divisor $\text{div}(\delta)$ associated to δ is defined by glueing the divisors $\text{div}(\alpha)$ on affine open sets. We see that the multiplication map $\cdot \delta : \mathcal{O}_X(\text{div}(\delta)) \rightarrow T_X$ defined by $h/\alpha \mapsto h \sum f_i \partial / \partial s_i$ induces an inclusion $\mathcal{O}_X(\text{div}(\delta)) \subset T_X$, and $\mathcal{O}_X(\text{div}(\delta))$ is an invertible 1-foliation of X . See [7], [3], [6], [8] for more details of the correspondences.

Next we recall the definition of global F-regularity.

Definition 1.2 ([10]). A projective variety over an F-finite field is *globally F-regular* if it admits some section ring that is F-regular.

For example, projective toric varieties are globally F-regular. In particular, projective spaces are globally F-regular. See [10], [9] for more examples and the general theory of globally F-regular varieties. In our situation, global F-regularity has a closed connection with splitting of Frobenius sandwiches.

Lemma 1.3 ([9], [8] Lemma 1.2). *Let X be a globally F-regular variety over k and Y be an F^e -sandwich of X with the finite morphism $\pi : X \rightarrow Y$ through which the Frobenius morphism of X factors. Then Y is globally F-regular if and only if the associated ring homomorphism $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_X$ splits as an \mathcal{O}_Y -module homomorphism.*

2. GLOBAL SECTIONS OF $T_{\mathbb{P}^n}$

Let $R = k[X_0, \dots, X_n]$, $X = \mathbb{P}^n = \text{Proj } R$, $S = H^0(X, \mathcal{O}_X(1)) = k\langle X_0, \dots, X_n \rangle$, $D_i = \partial / \partial X_i$, and $D_E = \sum_{i=0}^n X_i D_i \in \text{Der}_k R$. There exists an isomorphism f between k -modules $H^0(X, T_X)$ and $\bigoplus_{i=0}^n S D_i / (D_E)$ defined by the composition

$$H^0(X, T_X) \rightarrow \bigoplus_{i=0}^n S e_i / \left(\sum_{i=0}^n X_i e_i \right) \rightarrow \bigoplus_{i=0}^n S D_i / (D_E),$$

where the first map is induced by the Euler sequence and the second one is defined by $e_i \mapsto D_i$. Let $x_j = X_j / X_i$, $R_i = R_{(X_i)} = k[x_0, \dots, x_n]$, $U_i = D_+(X_i) = \text{Spec } R_i$, and $d_j = \partial / \partial x_j \in \text{Der}_k R_i$. Then

$$X_s D_i(X_t / X_i) = \begin{cases} -X_s X_t / X_i^2 = -x_s x_t & (s \neq i) \\ -X_t / X_i = -x_t & (s = i) \end{cases}$$

and

$$X_s D_j(X_t / X_i) = \begin{cases} 0 & (t \neq j) \\ X_s / X_i = x_s & (t = j) \end{cases}$$

for $j \neq i$. Hence

$$X_s D_i = \begin{cases} -x_s \sum_{t \neq i} x_t d_t & (s \neq i) \\ -\sum_{t \neq i} x_t d_t & (s = i) \end{cases}$$

and $X_s D_j = x_s d_j$ for $j \neq i$ on an open set $U \subset U_i$. We define the restriction map $\varphi_U : \bigoplus_{i=0}^n SD_i / (D_E) \rightarrow H^0(U, T_X)$ by $\overline{X_s D_i} \mapsto -x_s \sum_{t \neq i} x_t d_t$ ($s \neq i$), $\overline{X_i D_i} \mapsto -\sum_{t \neq i} x_t d_t$, and $\overline{X_s D_j} \mapsto x_s d_j$ ($j \neq i$) for the open set $U \subset U_i$. Then we have a commutative diagram

$$\begin{array}{ccccc} H^0(X, T_X) & \xleftarrow{\quad} & \bigoplus_{i=0}^n Se_i / \left(\sum X_i e_i \right) & \xrightarrow{\quad} & \bigoplus_{i=0}^n SD_i / (D_E) \\ \rho_{XU} \downarrow & & \downarrow & & \searrow \varphi_U \\ H^0(U, T_X) & \xleftarrow{\quad} & H^0(U, \text{Coker}(\mathcal{O}_X \rightarrow \mathcal{O}_X(1)^{n+1})) & & \end{array}$$

where ρ_{XU} is the restriction map of T_X . Therefore we have the following Lemma:

Lemma 2.1. *For any open set $U \subset U_i$, we have a commutative diagram*

$$\begin{array}{ccc} H^0(X, T_X) & \xrightarrow{f} & \bigoplus_{i=0}^n SD_i / (D_E) \\ \rho_{XU} \downarrow & \swarrow \varphi_U & \\ H^0(U, T_X) & & \end{array}$$

Lemma 2.2. *Let $\overline{D} \in \bigoplus_{i=0}^n SD_i / (D_E)$ and $\delta \in H^0(X, T_X)$ be the corresponding global section of T_X . Then $\text{Proj } R^D \cong X/\delta$.*

Proof. Let $F \in R$ be a homogeneous polynomial such that $F \in R^D$. Replacing F by $X_i^p F$, we may assume that $D_+(F) \subset U_i$. We can easily check that $(R_{(F)})^D = (R^D)_{(F)}$. For an open set $D_+(F) \subset \text{Proj } R^D$, we have $D_+(F) \cong \text{Spec } (R^D)_{(F)} \cong \text{Spec } (R_{(F)})^D$. On the other hand, X/δ is defined by glueing $D_+(F)/\delta \cong \text{Spec } (R_{(F)})^\delta \cong \text{Spec } (R_{(F)})^D$, since $\delta = D$ on $D_+(F) \subset U_i$ by Lemma 2.1. Therefore we have $\text{Proj } R^D \cong X/\delta$. \square

We define the p -th composition \overline{D}^p of $\overline{D} \in \bigoplus_{i=0}^n SD_i / (D_E)$ by \overline{D}^p . We say that $\delta \in H^0(X, T_X)$ (resp. $\overline{D} \in \bigoplus_{i=0}^n SD_i / (D_E)$) is p -closed if $\delta^p = \alpha\delta$ (resp. $\overline{D}^p = \alpha\overline{D}$) for some $\alpha \in k$.

Lemma 2.3. *Let $\overline{D} \in \bigoplus_{i=0}^n SD_i / (D_E)$ and $\delta \in H^0(X, T_X)$ be the corresponding global section of T_X . Supposed that $D \neq 0$. If δ is p -closed, then D is also p -closed.*

Proof. Let $K = \text{Frac} R$. If δ is p -closed, then \overline{D} is also p -closed. Hence $\overline{D}^p = \overline{D}^p = \alpha\overline{D} = \overline{\alpha D}$ for some $\alpha \in k$ and $D^p - \alpha D + \beta D_E = 0$ for some $\beta \in k$. Since $D \in \bigoplus_{i=0}^n SD_i$, we have $D^{p+1} - \alpha D^2 + \beta D = (D^p - \alpha D + \beta D_E) \circ D = 0$. This means that $t^{p+1} - \alpha t^2 + \beta t \in K^D[t]$ is divided by the minimal polynomial $\mu_D(t) \in K^D[t]$ of D , where we consider D as a K^D -linear map $K \rightarrow K$. Since $\mu_D(t) = t^{p^m} + \sum_{i=0}^{m-1} a_i t^i$ with $a_i \in R^D$ by [1] Lemma 2.4, we have $\mu_D(t) = t^p + a_0 t$. In particular, $D^p + a_0 D = 0$. Since $D \in \bigoplus_{i=0}^n SD_i$, we see that $a_0 \in k$. Therefore D is p -closed. \square

3. GLOBALLY F-REGULAR F-SANDWICHES OF DEGREE p OF \mathbb{P}^n

We will use the following lemmas in the proof of the main result.

Lemma 3.1. *Let X be a smooth variety, Y be a globally F-regular F-sandwich of degree p of X with the finite morphism $\pi : X \rightarrow Y$, and $L \subset T_X$ be the corresponding 1-foliation. Then we have*

$$\text{Hom}_{\mathcal{O}_Y}(\pi_* \mathcal{O}_X, \mathcal{O}_Y) \cong H^0(Y, \pi_*(L^{\otimes(p-1)})) = H^0(X, L^{\otimes(p-1)}).$$

Proof. See [4] Theorem 3.4. \square

Lemma 3.2. *Let $\mathfrak{m} = (X_0, \dots, X_n)$ be the maximal ideal of $R = k[X_0, \dots, X_n]$, and $D \in \bigoplus_{i=0}^n SD_i$ be a p -closed derivation. Supposed that D is not nilpotent. Then there exists $D' = \sum_{i=0}^n a_i D_i \in \text{Der}_k R$ with $a_i \in \mathbb{F}_p$ such that $R^D \cong R^{D'}$.*

Proof. Since $D \in \bigoplus_{i=0}^n SD_i$ and D is not nilpotent, we have $D^p = \alpha D$ for some $\alpha \in k^\times$. Replacing D by $\alpha^{1/(1-p)} D$, we may assume that $D^p = D$. We define $\overline{D} \in \text{Der}_k(\mathfrak{m}/\mathfrak{m}^2)$ by $\overline{D}(f) = \overline{D(f)}$. Since $D^p - D = 0$, the minimal polynomial $\mu_{\overline{D}}(t) \in k[t]$ of \overline{D} divides $t^p - t = t(t-1)(t-2)\cdots(t-(p-1))$. Hence \overline{D} is diagonalizable with eigenvalues $a_0, \dots, a_n \in \mathbb{F}_p$. Let Y_0, \dots, Y_n be elements of S such that $\overline{Y}_0, \dots, \overline{Y}_n \in \mathfrak{m}/\mathfrak{m}^2$ are linearly independent eigenvectors of \overline{D} corresponding to eigenvalues a_0, \dots, a_n , respectively. Then $\overline{D(Y_i)} = \overline{D(Y_i)} = a_i \overline{Y_i} = \overline{a_i Y_i}$. Since $D \in \bigoplus_{i=0}^n SD_i$ and $Y_i \in S$, we have $D(Y_i) \in S$. Thus $D(Y_i) = a_i Y_i$. After a change of coordinates $X_i \mapsto Y_i$, we have $D = \sum_{i=0}^n a_i Y_i \partial / \partial Y_i$. This completes the proof. \square

Lemma 3.3. *Let $\overline{D} = \overline{\sum_{i=0}^n a_i X_i D_i} \in \bigoplus_{i=0}^n SD_i / (D_E)$ with $a_i \in \mathbb{F}_p$. Supposed that $\overline{D} \neq 0$. Then $\text{Proj } R^D$ is a toric variety.*

Proof. We refer to [2] for the general theory of toric varieties.

Let $\delta \in H^0(\mathbb{P}^n, T_{\mathbb{P}^n})$ be the corresponding global section of $T_{\mathbb{P}^n}$. We have $\overline{D} = \overline{\sum_{i=1}^n (a_i - a_0) X_i D_i}$ and $a_i - a_0 \neq 0$ for some i . Replacing $a_1 - a_0$ by $a_i - a_0$, and multiplying \overline{D} by $(a_1 - a_0)^{-1}$, we may assume that $\overline{D} = \overline{X_1 D_1 + \sum_{i=2}^n a_i X_i D_i}$. Then we have $\delta|_{U_0} = \overline{D}|_{U_0} = x_1 d_1 + \sum_{i=2}^n a_i x_i d_i$, which is a description of δ as a rational vector field.

Let $N = \mathbb{Z}^n$ be a lattice, M be the dual lattice of N , and Σ be the fan in $N \otimes \mathbb{R}$ corresponding to the projective space \mathbb{P}^n . Let $N' = N + \mathbb{Z} \frac{1}{p}(1, a_2, \dots, a_n)$ be an overlattice of N , and M' be the dual lattice of N' . We have

$$\begin{aligned}
M' &= (N')^\vee \\
&= \left(\mathbb{Z} \frac{1}{p}(1, a_2, \dots, a_n) \oplus \mathbb{Z} \frac{1}{p}(1, a_2 - p, \dots, a_n) \oplus \cdots \oplus \mathbb{Z} \frac{1}{p}(1, a_2, \dots, a_n - p) \right)^\vee \\
&= \mathbb{Z}(p - a_2 - \cdots - a_n, 1, \dots, 1) \\
&\quad \oplus \mathbb{Z}(a_2, -1, 0, \dots, 0) \oplus \cdots \oplus \mathbb{Z}(a_n, 0, \dots, 0, -1) \\
&= \mathbb{Z}(p, 0, \dots, 0) \oplus \mathbb{Z}(a_2, -1, 0, \dots, 0) \oplus \cdots \oplus \mathbb{Z}(a_n, 0, \dots, 0, -1) \\
&= \left\{ (s_1, \dots, s_n) \in M \left| s_1 + \sum_{i=2}^n a_i s_i \equiv 0 \pmod{p} \right. \right\} \subset M.
\end{aligned}$$

Let σ_i be the cone corresponding to $U_i = D_+(X_i) \subset \mathbb{P}^n$. Since $(R_i)^\delta = R_i \cap K(\mathbb{P}^n)^\delta = k[(\sigma_i)^\vee \cap M']$, we see that \mathbb{P}^n/δ is the toric variety whose corresponding fan is Σ in $N' \otimes \mathbb{R}$. Since $\mathbb{P}^n/\delta \cong \text{Proj } R^D$ by Lemma 2.2, $\text{Proj } R^D$ is the toric variety. \square

Theorem 3.4. *Globally F -regular F -sandwiches of degree p of \mathbb{P}^n are toric varieties.*

Proof. Let Y be a globally F -regular F -sandwich of degree p of \mathbb{P}^n with the finite morphism $\pi : \mathbb{P}^n \rightarrow Y$ through which the Frobenius morphism of \mathbb{P}^n factors, and let $L \subset T_{\mathbb{P}^n}$ (resp. $\delta \in \text{Der}_k K(\mathbb{P}^n)$) be the corresponding 1-foliation (resp. the p -closed rational vector field). Since the associated ring homomorphism $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_{\mathbb{P}^n}$ splits by Lemma 1.3, there is a nonzero \mathcal{O}_Y -module homomorphism $\pi_* \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_Y$. By Lemma 3.1, $L^{\otimes(p-1)}$ has a nonzero global section, and so does L . Let $\delta = \alpha \sum f_i \partial / \partial s_i$ be a local expression of δ , where s_i are local coordinates, f_i are regular functions without common factors, and $\alpha \in K(X)$. Multiplying δ by a suitable rational function, we may

assume that α are regular functions, since $\mathcal{O}_X(\operatorname{div}(\delta)) \cong L$. Then δ is a global section of T_X . Let \overline{D} be the corresponding element of $\bigoplus_{i=0}^n SD_i / (D_E)$. Since δ is p -closed, D is also p -closed by Lemma 2.3. Then there exists $D' = \sum a_i X_i D_i \in \operatorname{Der}_k R$ with $a_i \in \mathbb{F}_p$ such that $R^D \cong R^{D'}$ by Lemma 3.2. We have $Y \cong \mathbb{P}^n / \delta \cong \operatorname{Proj} R^D \cong \operatorname{Proj} R^{D'}$ by Lemma 2.2 and Lemma 3.2. Therefore we see that the F -sandwich Y is a toric variety by Lemma 3.3. \square

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